The MacDonald Equation

Freeman Dyson

The MacDonald Equation is the most beautiful thing that I ever discovered. It belongs to the theory of numbers, the most useless and ancient branch of mathematics. My friend Ian MacDonald had the joy of discovering it first, and I had the almost equal joy of discovering it second. Neither of us knew that the other was working on it. We had daughters in the same class at school, so we talked about our daughters and not about mathematics. We discovered an equation for the "Tau-function" (written $\tau(n)$ in the equation), an object explored by the Indian genius Srinivasa Ramanujan four years before he died at age thirty-two. Here I wrote down MacDonald's equation for the Tau-function. The MacDonald equation has an amazing five-fold symmetry that Ramanujan missed. You can see the five-fold symmetry in the ten differences multiplied together on the right-hand side of the equation. We are grateful to Ramanujan, not only for the many beautiful things that he discovered, but also for the beautiful things that he left for other people to discover.

To explain how the MacDonald equation works, let us look at the first three cases, *n=1, 2, 3*. The sum is over sets of five integers *a, b, c, d, e* with sum zero and with the sum of their squares equal to *10n*. The "(*mod 5*)" statement means that *a* is of the form *5j+1*, *b* is of the form *5k+2*, and so on up to *e* of the form *5p+5*, where *j*, *k*, and *p* are positive or negative integers. The exclamation marks in the equation mean *1!=1*, *2!=1×2=2*, *3!=1×2×3=6*, *4!=1×2×3×4=24*. So when *n=1*, the only choice for *a, b, c, d, e* is *1, 2, -2, -1, 0*, and we find *tau*(*1*)*=1*. When *n=2*, the only choice is *1, -3 ,3 ,-1 , 0*, and we find *tau*(*2*)*=-24*. When *n=3*, there are two choices, *1, -3, -2, 4, 0* and *-4, 2, 3, -1, 0,* which give equal contributions, and we find *tau*(*3*)*=252*. It is easy to check that these three values of *tau*(*n*) agree with the values given by Ramanujan's equation.

The MacDonald equation is a special case of a much deeper connection that Ian MacDonald discovered between two kinds of symmetry which we call modular and affine. The two kinds of symmetry were originally found in separate parts of science, modular in pure mathematics and affine in physics. Modular symmetry is displayed for everyone to see in the drawings of flying angels and devils by the artist Mauritz Escher. Escher understood the mathematics and got the details right. Affine symmetry is displayed in the peculiar groupings of particles created by physicists with high-energy accelerators. The mathematician Robert Langlands was the first to conjecture a connection between these and other kinds of symmetry. Ian MacDonald took a big step toward making Langlands's dream come true. The equation that I wrote down here is a small piece of MacDonald's big step.