Ampère's Law
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Much of my research involves the interaction between certain topics in differential geometry, related to Mathematical Physics, and the *topology* of four-dimensional spaces. This blackboard represents some of these ideas, partly by analogy in three dimensions.

The main theme of the blackboard is Ampère's Law in electromagnetism, and much of the board is similar to what one sees in standard physics texts. The picture in the upper-left side depicts a current **j** flowing in a closed wire, indicated by the thick black curve. The current produces a magnetic field **B**, indicated by the small arrows. In a two-dimensional situation this corresponds to the pattern you would see by scattering iron filings on a sheet of paper. The magnetic field is defined everywhere, so really we should imagine a little arrow at each point, but it is only practical to draw a few of them. The basic physical phenomenon, stated in ordinary language, is that the magnetic field "goes around" the wire, and Ampère's Law amounts to a precise quantitative expression of this idea.

The notion of a "vector field" such as the magnetic field (or the current, which is a vector field confined to the interior of the wire) was a crucial conceptual advance in early 19th century Mathematical Physics. It gives a common framework for describing electricity, magnetism, gravity and much else. One important concept is the "flux" of a vector field through a surface. The mathematical definition involves the process of integration over the surface, but the idea can be made intuitive by imagining that the vector field represents the velocity of a fluid: then the flux is the rate of flow of fluid through the surface.

One formulation of Ampère's Law – the "integral form" – states that the total circulation of the current around the boundary of a piece of surface is equal to the flux of the current through the surface. This is illustrated by the disc cutting across the wire in the middle of the blackboard. Another formulation – the "differential form" – is the set of the equations at the lower-right of the board, expressing the components of the current in terms of derivatives in the three space directions x, y, z of the components of the magnetic field.

This board attempts to convey, or at least hint at, several broad aspects of mathematics which I find beautiful. On the left we see a picture and words: in the lower right, a set of equations. These are different descriptions of the same thing and they stimulate different ways of thought: pictorial and symbolic. Further, the picture can represent an actual object – a copper wire carrying a current – in the real world, but a mathematician will often draw such a picture with a schematic meaning: for example, we might imagine that rather than a one-dimensional wire in three-dimensional space we are looking at a three-dimensional object in seven-dimensional space, or even objects in an infinite-dimensional space. Such extensions of our physical intuition to more abstract situations can be remarkably effective. The interactions between the intuitive, the pictorial, the symbolic and the abstract are beautiful and delight the mind.

What does this have to do with *topology* – the study of phenomena independent of continuous deformations? This is indicated schematically by the fact that the wire loop is *knotted* – it cannot be deformed into a standard circle without cutting and re-joining. This is something which is not so easy to demonstrate mathematically but which we understand from our experience. Further, we understand that such knots could be arbitrarily complicated, providing an intuitive demonstration that topology can be a subtle matter. At a more detailed level, there are precise connections between knots and four-dimensional spaces: a knot encodes instructions for building a four-dimensional space by gluing together some standard building blocks.

The mathematics which this board alludes to is represented in spirit rather than precision. The idea which it seeks to convey is that doing "something like" studying the magnetic field generated by a knotted current could have "something to do" with the topology of knots and four-dimensional spaces. Over the past three decades there have indeed been many developments in this spirit, although the detail is somewhat different. For example, these developments involve generalisations of electromagnetism to "Yang-Mills fields" and are also bound up with Quantum Mechanics and Quantum Field Theory. These last are represented in the lower-left of the board, where we consider the flux of the magnetic field through a small disc. This quantity does not have any meaning in classical electromagnetism (as far as the writer knows), but is central in the quantum theory of the interaction of the magnetic field with the "wave function" of an electron.

What is special about three and four dimensions? In topology, this is a profound question. It turns out that spaces of dimension bigger than four are in many respects easier to understand. Even without knowing any of their detailed meaning, one can see some aspect of this dimension-specificity from the displayed formulae. They are given by permuting in cyclic order the three coordinates x, y, z. The fact that one can write down these equations depends on the fact that there are exactly three pairs (xy), (yz), (zx) of three objects x, y, z. One can generalise electromagnetic theory to higher dimensions but then the magnetic field is no longer a vector field but a more complicated kind of object. Three dimensions is special because the magnetic field is a vector field, just like the electric field. This extends to similar phenomena in four dimensions which are somehow bound up with the special topological features. Understanding all of this, at a fundamental level, is a fascinating problem and we only see at the moment some shadow of the ultimate truth. Here we find other aspects of the beauty of mathematics: surprising but mysterious connections between different fields, and the intermingling of the seemingly-simple and well-understood with the completely unknown.